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THE EFFECT OF TRANSVERSE SHEAR ONTO THE SINGULAR SOLUTIONS OF A COMPLETE SPHERICAL SHELL†

NIKOLAOS SIMOS
Earthquake Research Center, City College of the City University of New York, New York, NY 10031, U.S.A.

and

ALI M. SADEGH
Mechanical Engineering Department, The City College of the City University of New York, New York, NY 10031, U.S.A.

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Abstract—Green's functions for the field variables of a complete sphere subjected to normal surface traction are obtained with "free space" properties. Further, self-equilibrated singular solutions of the variables associated with tangentially applied point loads and concentrated surface moments are constructed. The solution formulae are derived within the framework of the improved theory of thin shells and thus incorporate the effect of transverse shear in the equilibrium of the shell element. Despite the complex character of the solution, expressed in terms of complex Legendre functions, the closed form of it reveals the effects of the new assumptions (presence of shear strains) onto the singular behavior of the associated kernels. Numerical results for the field variables demonstrate the differences between the two theories, classical and improved.

INTRODUCTION

The complexity of the analysis associated with the study of shells has over the years been dealt with the introduction of suitable auxiliary variables into the governing differential equations. The advantage of such approach is that the system is reduced to a set of uncoupled and/or coupled equations which are easier to deal with. In the improved theory of shells the effect of the transverse shear deformation is included in the analysis. Even though the basic equations of equilibrium are the same as in the classical theory, the independent role of the angular rotations of the normal, β_i, introduces two additional variables which in turn upgrade the order of the governing differential system. Furthermore, the shearing stress resultants, Q_i, are the direct effect of the nonvanishing shear stress and no longer a requirement for the overall equilibrium of the shell element.

The incorporation of the shear effect has been introduced in the analysis of plates by Reissner (1947) and later by Naghdi (1956) in the deductions of differential equations for thin elastic shells. The reduction of the primary system of equations of a shallow spherical shell has been obtained by Kalnins (1961). A similar approach has been used by Prasad (1964) in the derivation of a system of coupled equations in the transverse displacement W and a set of suitable auxiliary variables of a nonshallow spherical shell. In the works of Wilkinson and Kalnins (1966a, b) an equivalent system of equations to that in Prasad (1964) was obtained. In their detailed analysis the nonsymmetric dynamic problem of an open spherical shell was studied and results were obtained for the shell response under the action of a horizontal force and moment. In Wilkinson and Kalnins (1966b) an exact solution for the Green's function, represented by the solution to an arbitrarily located normal concentrated load acting on an open spherical shell was derived. The effect of transverse shear has also been considered by Delale and Erdogan (1979) in their study of a cracked shallow spherical cap.

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The aim of this paper is the derivation of exact, closed form solutions of the Green's functions of the "free space" type. Such form of the Green's function will be associated with the complete sphere and it will be required to satisfy the overall equilibrium. Green's solutions have been obtained by Simmonds (1968) and Koiter (1963) for the nonshallow classical theory of shells. Within the framework of the improved theory, Green's functions have been presented by Nordgren (1963) for the thermoelastic problem of shallow shells. Further, the scope of this analysis is to introduce singular solutions of closed form for concentrated tangential loads and moments which would apply in a self-equilibrating fashion. In the case of the "free space" Green's function the unit concentrated normal load is applied in the form of a Dirac Delta distribution accompanied by an axisymmetric normal surface traction which ensures overall equilibrium. As has been discussed in Simos (1988), the "physical" role of the distributed surface traction is mathematically compatible with the whole idea of free space Green's function and it obtains its form directly from the governing differential equation. The equilibrium requirement in the case of the unit tangential load is satisfied with the application of a similar load at the opposite pole and in the opposite direction together with a concentrated moment to eliminate the resulting couple. The closed form solutions are in terms of complex Legendre functions and elementary functions. Such a representation appears suited to numerical evaluation. The singular formulae expressions are evaluated in the vicinity of the pole with the help of the expansions of the Legendre functions in the neighborhood of such point and comparisons are performed with the analogous problems of the classical theory of shells. The most striking differences in the character of the singularities are observed in the transverse displacement W of the normal point load and the shear resultant of the tangential load and moment. The character of the singularities is also compared with that of previous solutions.

Lastly, results of a number of shell problems are presented in comparison format with the corresponding problems of the classical theory. These results show not only the effective point load neighborhood where transverse shear is important but also serve as justification of the derived expressions of the shell variables.

SYSTEM OF GOVERNING EQUATIONS

According to the analysis performed by Prasad (1964), the governing equations of a nonshallow spherical shell of middle surface radius R, Young's modulus E and Poisson's ratio μ and thickness h are uncoupled with the introduction of the auxiliary variables U, Ψ, Γ and Λ which relate to the displacement vector as well as the angular rotations of the normal. In the analysis, the primary system of equations is reduced to a secondary system of equations in terms of the new variables and the transverse displacement function W. This secondary system, for the case of only normal surface traction q_n being present, is uncoupled into a set of two differential equations:

$$\nabla^6 W + p_4 \nabla^4 W + p_2 \nabla^2 W + p_0 W + L_0[q_n] = 0 \tag{1}$$

where

$$p_4 = 3-\mu-k_s(1-\mu^2), \quad p_2 = \frac{1-\mu^2}{\xi} + 2(1-\mu) + k_s(1-\mu^2)(\mu-3),$$

$$p_0 = \frac{2(1-\mu^2)}{\xi} - 2(1-\mu)(1-\mu^2)k_s, \quad L_0 = \frac{R^4}{D}(\nabla^2+1-\mu)\left[1 - \frac{k_s(1-\mu^2)D}{EhR}(\nabla^2+1-\mu)\right],$$

$$k_s = \frac{2k_s^0}{1-\mu}, \quad \xi = \frac{h^2}{12R^2}$$

and

$$(\nabla^2+2)\left[\nabla^2+2-\frac{1}{\xi k_s^0}\right]\Psi = 0. \tag{2}$$

$k_s^0 = \frac{5}{6}$ corresponds to the coefficient of shear as defined in [1] and [2]. The remaining auxiliary variables Λ, U and Γ are governed by the system of equations (A6)–(A10) in the Appendix. Evaluation of W and Ψ will lead to the determination of the complete set of the system variables.

FUNDAMENTAL SOLUTIONS OF NORMAL SURFACE TRACTION

In constructing the fundamental singular solutions of the operators in (1) and (2) we should require that such solutions satisfy not only the differential operators but also demonstrate singular behavior at a unique point on the complete sphere. Also, with the same argument used in Simos (1988) for the classical theory, the condition of equilibrium in the vicinity of the pole and the vanishing of the tangential displacement and the rotation vector at the pole must be satisfied by the fundamental solution without being introduced *a priori*.

Consider the self-equilibrated normal surface traction q_n applied over the middle surface of a complete sphere and expressed in the axisymmetric form

$$q_n = \frac{1}{R^2}\left[\frac{\delta(\gamma)}{2\pi \sin\gamma} - \frac{3}{4\pi}\cos\gamma\right] \tag{3}$$

where $\delta(\gamma)/R^2 2\pi \sin\gamma$ is the Dirac Delta function distribution applied at the pole $\gamma = 0$ of the rotated geographical coordinate (γ, η) and $-(3/R^2 4\pi)\cos\gamma$ is the axisymmetric distributed surface traction introducing a resultant equal and opposite to that of the Delta function. Such surface traction is applied to satisfy the overall equilibrium of the complete sphere. As it was shown for the classical shell theory the participation and form of the surface traction is a direct result of the particular form of the differential operator in (1). The component ∇^2+2 of the operator has no homogeneous solution that presents singular behavior at only one point on the unit sphere. However, there exists a generalized Green's function for the operator ∇^2+2, which indeed is singular at only one point, that is the solution of ∇^2+2 with nonhomogeneous part the expression in (3). The relation between the two surface coordinate systems (ϕ, θ) and (γ, η), can be viewed through the identity

$$\cos\gamma = \cos\phi\cos\phi' + \sin\phi\sin\phi'\cos(\theta-\theta') \tag{4}$$

where (ϕ, θ) represents an arbitrary surface point and (ϕ', θ') the point where the Delta function applies ($\gamma = 0$).

Consider (1) in the form

$$L_1[W] = L_0[q_n] \tag{5}$$

where

$$L_1 = \nabla^6 + p_4\nabla^4 + p_2\nabla^2 + p_0. \tag{6}$$

We can express operator L_1 in its equivalent form

$$L_1 = (\nabla^2 - r_1)(\nabla^2 - r_2)(\nabla^2 - r_3) \tag{7}$$

provided that r_i $(i = 1, 2, 3)$ are the roots of the cubic equation

$$r_i^3+p_4r_i^2+p_2r_i+p_0 = 0. \tag{8}$$

With one real root, r_1, and two complex conjugate roots, r_2 and r_3, we express L_1 in the form

$$L_1 = (\nabla^2+2)[\nabla^2+\nu(\nu+1)][\nabla^2+\lambda(\lambda+1)] \tag{9}$$

where

$$r_1 = -2, \quad -r_2 = \nu(\nu+1), \quad -r_3 = \lambda(\lambda+1).$$

Obviously ν and λ are complex conjugate parameters.

The component ∇^2+2 of the operator L_1 represents the membrane solution of a loaded spherical shell. It is interesting to note that for both theories, classical and improved, the contribution of the membrane solution is somewhat decoupled from the bending solution which dominates in the vicinity of the pole.

The complete form of (1) can now be expressed as

$$(\nabla^2+2)[\nabla^2+\nu(\nu+1)][\nabla^2+\lambda(\lambda+1)]W(\mathbf{x};\mathbf{x}')$$
$$= \frac{R^2}{D}(\nabla^2+1-\mu)\left(1-\frac{k_s(1-\mu^2)D}{ER^2}(\nabla^2+1-\mu)\right)\left[\frac{\delta(\gamma)}{2\pi\sin\gamma}-\frac{3}{4\pi}\cos\gamma\right] \tag{10}$$

where

$$\nabla^2 = \frac{\mathrm{d}^2}{\mathrm{d}\gamma^2}+\cot\gamma\,\frac{\mathrm{d}}{\mathrm{d}\gamma}.$$

We express the fundamental solution of the transverse displacement $W(\mathbf{x};\mathbf{x}')$ in the equivalent form

$$W(\mathbf{x};\mathbf{x}') = W_1(\mathbf{x};\mathbf{x}')+W_2(\mathbf{x};\mathbf{x}') \tag{11}$$

such as

$$L_1[W_1(\mathbf{x};\mathbf{x}')] = L_0[\delta(\mathbf{n}-\mathbf{n}')], \quad L_1[W_2(\mathbf{x};\mathbf{x}')] = L_0\left[-\frac{1}{R^2}\frac{3}{4\pi}\cos\gamma\right]. \tag{12}$$

After operating onto the right-hand side of the equation governing W_2 we seek the particular integral of

$$(\nabla^2+2)[\nabla^2+\nu(\nu+1)][\nabla^2+\lambda(\lambda+1)]W_2(\mathbf{x};\mathbf{x}') = 3C_0'\cos\gamma \tag{13}$$

where

$$C_0' = \frac{3R^2(1+\mu)[1+C_0(1+\mu)]}{4\pi D}, \quad C_0 = \frac{k_s(1-\mu^2)D}{ER^2}.$$

We introduce the function $f_p(\mathbf{x};\mathbf{x}')$ such as

$$(\nabla^2+2)f_p(\mathbf{x};\mathbf{x}') = 3C_0'\cos\gamma \tag{14}$$

which leads to

$$f_p(\mathbf{x};\mathbf{x}') = -C_0'[1+\cos\gamma\ln(1-\cos\gamma)]. \tag{15}$$

We express $W_2(\mathbf{x};\mathbf{x}')$ in the form

$$W_2(\mathbf{x}; \mathbf{x}') = A_p f_p(\mathbf{x}; \mathbf{x}') \tag{16}$$

and introduce it into (13) from which we finally obtain

$$A_p = \frac{1}{[\nu(\nu+1)-2][\lambda(\lambda+1)-2]}. \tag{17}$$

Thus, according to (16) we can write

$$W_2(\mathbf{x}; \mathbf{x}') = -T_1[1+\cos\gamma \ln(1-\cos\gamma)] \tag{18}$$

where

$$T_1 = ([\mathrm{Re}\,\{\nu(\nu+1)-2\}]^2+[\mathrm{Im}\,\{\nu(\nu+1)-2\}]^2)^{-1}\cdot C_0'.$$

We now consider the differential operator which governs $W_1(\mathbf{x}; \mathbf{x}')$ and which we express in the form

$$W_1(\mathbf{x}; \mathbf{x}') = L_0[U_s(\mathbf{x}; \mathbf{x}')] \tag{19}$$

provided that the introduced scalar function $U_s(\mathbf{x}; \mathbf{x}')$ satisfies

$$(\nabla^2+2)[\nabla^2+\nu(\nu+1)][\nabla^2+\lambda(\lambda+1)]U_s(\mathbf{x}; \mathbf{x}') = \frac{\delta(\gamma)}{2\pi R^2 \sin\gamma}. \tag{20}$$

By utilizing the argument according to which the fundamental singularity can only be singular at only one point over the domain of interest, we retain only those independent solutions of the homogeneous form of (20) that satisfy the above requirement and write the scalar function U_s in the form

$$U_s(\mathbf{x}; \mathbf{x}') = A_2 P_\nu(-\cos\gamma)+A_3 P_\lambda(-\cos\gamma) \tag{21}$$

where A_2 and A_3 are complex conjugate arbitrary constants. The evaluation of the constants is dictated by the requirement that the particular solution of U_s satisfies (20) in the vicinity of the pole $\gamma = 0$.

After expressing (20) in the form

$$\nabla^2[(\nabla^2+2)[\nabla^2+\lambda(\lambda+1)]U_s(\mathbf{x}; \mathbf{x}')] = \frac{\delta(\gamma)}{2\pi R^2 \sin\gamma} - \nu(\nu+1)U_s(\mathbf{x}; \mathbf{x}') \tag{22}$$

we utilize the divergence theorem,

$$\iint_\sigma \nabla^2 f(\mathbf{x}; \mathbf{x}')\, \mathrm{d}\sigma = \int_C \frac{\partial f(\mathbf{x}; \mathbf{x}')}{\partial \mathbf{n}}\, \mathrm{d}C$$

which can simply be formulated around a circular contour that encloses the pole $\gamma = 0$. It is apparent from (23) that only the constant A_2 remains in the left-hand side of the expression, while in the limit as $\gamma \to 0$ the surface integral of the right-hand side is equal to 1. According to the limiting values of the Legendre functions, (22) leads to the evaluation of A_2 and consequently A_3.

$$A_2 = \frac{1}{4[\nu(\nu+1)-\lambda(\lambda+1)][2-\nu(\nu+1)]\sin\nu\pi}, \quad A_3 = \mathrm{CONJG}\,[A_2]. \tag{23}$$

Finally $W_1(\mathbf{x};\mathbf{x}')$ can be written in the form

$$W_1(\mathbf{x};\mathbf{x}') = \frac{R^2}{D}[(\nabla^2+1-\mu)-C_0'(\nabla^2+1-\mu)^2]U_s(\mathbf{x};\mathbf{x}') = 2\,\mathrm{Re}\,(A_n P_\nu(-\cos\gamma)) \tag{24}$$

where

$$A_n = \frac{R^2}{D}([1-\mu-\nu(\nu+1)]A_2[1-C_0'[1-\mu-\nu(\nu+1)]]).$$

The complete expression of the fundamental solution of the displacement function W will take the form of the sum of the solutions described by (18) and (24). However, in order for the combined expression to correspond to the Generalized Green's function of the spherical domain, due to its association with the equilibrating surface traction, the orthogonality condition

$$\iint_\sigma W_2(\mathbf{x};\mathbf{x}')\cos\gamma\,d\sigma = 0 \tag{25}$$

must be met. The above requirement leads to

$$W_2 = -T_1[1+\cos\gamma\ln[1-\cos\gamma]+B\cos\gamma]$$

where

$$B = \tfrac{4}{3}-\ln 2. \tag{26}$$

The remaining auxiliary variables Ψ, Λ, U and Γ are evaluated with the help of the system of equations (A6)–(A10) together with the incorporation of the displacement function W derived above. We recall that the secondary system of equations was expressed in terms of the functions W and Ψ and we observe that the auxiliary variable Ψ has no dependence on the normal surface traction q_n. For the axisymmetric case the general homogeneous solution of (2) will take the form

$$\Psi(\mathbf{x};\mathbf{x}') = A_1^\psi P_1(\cos\gamma)+A_2^\psi Q_1(\cos\gamma)+B_1^\psi P_\omega(\cos\gamma)+B_2^\psi Q_\omega(\cos\gamma) \tag{27}$$

where

$$\omega(\omega+1) = 2-\frac{1}{\xi k_s^0} \quad \text{and} \quad A_1^\psi, A_2^\psi, B_1^\psi, B_2^\psi$$

are arbitrary constants. The auxiliary variable $\Lambda(\mathbf{x};\mathbf{x}')$ can be deduced from

$$\Lambda(\mathbf{x};\mathbf{x}') = -k_s^0(\nabla^2+2)\Psi(\mathbf{x};\mathbf{x}') = -k_s^0[2-\omega(\omega+1)][B_1^\psi P_\omega(\cos\gamma)+B_2^\psi Q_\omega(\cos\gamma)]. \tag{28}$$

The requirement, however, of a single singularity in the domain and the elimination of the regular solutions from the expressions for Ψ and Λ yields

$$\Psi(\mathbf{x};\mathbf{x}') = B_1^\psi P_\omega(-\cos\gamma) \tag{29}$$

and

$$\Lambda(\mathbf{x};\mathbf{x}') = -k_s^0[2-\omega(\omega+1)]B_1^\psi P_\omega(-\cos\gamma). \quad (30)$$

The incorporation of the final forms of $\Psi(\mathbf{x};\mathbf{x}')$ and $\Lambda(\mathbf{x};\mathbf{x}')$ into the singular solution would require the evaluation of the arbitrary constant B_1^ψ, which in turn would have to satisfy a requirement set *a priori*. But the only condition the variable Ψ must satisfy, besides being a solution of the governing system, is the existence of the Delta function, to which it apparently has no relation since it is independent of the normal surface traction. We observe, however, that variables Ψ and Λ are related to the tangential displacement u_γ, the rotation of the normal and consequently to the shearing resultant Q_γ. The axisymmetry of the solution reasonably requires that when approaching the pole $\gamma = 0$, the two vectors u_γ and β_γ satisfy

$$\lim_{\gamma\to 0} u_\gamma = \lim_{\gamma\to 0}\left[\frac{\mathrm{d}U}{\mathrm{d}\gamma} - \Psi R\sin\gamma\right] = 0, \quad \lim_{\gamma\to 0}\beta_\gamma = \lim_{\gamma\to 0}\left[\frac{\mathrm{d}\Gamma}{\mathrm{d}\gamma} - \Lambda\sin\gamma\right] = 0. \quad (31)$$

The general expressions for Ψ and Λ contain a logarithmic singularity and their contribution to the limiting value of u_γ and β_γ vanishes since

$$\lim_{\gamma\to 0}[\sin\gamma\Psi] = \lim_{\gamma\to 0}[\sin\gamma\Lambda] = 0. \quad (32)$$

Thus, for the axisymmetric case and without any loss of true representation of the system's variables, we can set

$$\Psi(\mathbf{x};\mathbf{x}') = \Lambda(\mathbf{x};\mathbf{x}') = 0. \quad (33)$$

With the elimination of Ψ and Λ from the secondary system of equations, the remaining variables U, Γ are expressed in terms of W and the surface traction q_n. Thus we can write

$$\Gamma(\mathbf{x};\mathbf{x}') = -\left[\frac{\xi^2k_s^2}{RD_3}\nabla^4 + \frac{\xi k_s D_1}{RD_3}\nabla^2 + \frac{D_2}{RD_3}\right]W(\mathbf{x};\mathbf{x}') - \frac{\xi^2k_s^2}{RD_3}(\nabla^2+1-\mu)\left[\frac{R^2(1-\mu^2k_s)}{Eh}q_n\right] \quad (34)$$

where

$$D_1 = 1+2\xi k_s - \xi k_s^2(1-\mu^2), \quad D_2 = 1+(1+\mu)\xi k_s - 2(1-\mu^2)\xi^2k_s^2,$$
$$D_3 = 1+2\mu\xi k_s + (\mu^2-1)\xi^2k_s^2.$$

We write $\Gamma(\mathbf{x};\mathbf{x}')$ in the form

$$\Gamma(\mathbf{x};\mathbf{x}') = \Gamma_s(\mathbf{x};\mathbf{x}') + \Gamma_r(\mathbf{x};\mathbf{x}') \quad (35)$$

where the subscripts s and r identify the singular and the regular components of Γ. The regular solution Γ_r is to be derived as the dependence of Γ onto the equilibrating surface traction $q_{n'}$, while Γ_s will be associated with the fundamental singular solution of the transverse displacement W. Execution of the operators leads to:

$$\Gamma(\mathbf{x};\mathbf{x}') = 2\,\mathrm{Re}\,(A_n^\Gamma P_v(-\cos\gamma)) + T_1^\Gamma[1+\cos\gamma\ln(1-\cos\gamma)] + C^\Gamma\cos\gamma \quad (36)$$

where

$$A_n^{\Gamma} = -\left[\frac{\xi^2 k_s^2}{RD_3}[\nu(\nu+1)]^2 - \nu(\nu+1)\frac{\xi k_s D_1}{RD_3} + \frac{D_2}{RD_3}\right]A_n,$$

$$T_1^{\Gamma} = \left[4\frac{\xi^2 k_s^2}{RD_3} - 2\frac{\xi k_s D_1}{RD_3} + \frac{D_2}{RD_3}\right]T_1$$

$$C^{\Gamma} = \left[\frac{\xi^2 k_s^3 R(1+\mu)(1-\mu^2)C_q}{D_3 Eh} + \frac{2\xi k_s D_1 - 4\xi^2 k_s^2 - D_2}{RD_3}C_p\right],$$

$$C_q = -\frac{3}{4\pi R^2}, \quad C_p = -T_1 B.$$

Similar procedures to the ones above are used to determine the remaining variable $U(\mathbf{x};\mathbf{x}')$ which is finally written in the form

$$U(\mathbf{x};\mathbf{x}') = -(\xi R)\Gamma(\mathbf{x};\mathbf{x}') + \left((1-2\xi k_s) + \frac{\xi[2-(1-\mu^2)k_s]}{1-\mu^2}\nabla^2 + \frac{\xi}{1-\mu^2}\nabla^4\right)W(\mathbf{x};\mathbf{x}') - \frac{R^2}{Eh}q_n + \frac{\xi k_s R^2}{Eh}(\nabla^2 + 1 - \mu)q_n \quad (37)$$

and which, after the execution of the operations, leads to

$$U(\mathbf{x};\mathbf{x}') = 2\ \mathrm{Re}\ (A_n^U P_\nu(-\cos\gamma)) + T_1^U[1+\cos\gamma\ \ln\ (1-\cos\gamma)] + C^U\cos\gamma \quad (38)$$

where

$$A_n^U = -\xi R A_n^{\Gamma} + \left(1 - 2\xi k_s - \frac{\xi[2-(1-\mu^2)k_s]}{(1-\mu^2)}\nu(\nu+1) + \frac{\xi}{1-\mu^2}[\nu(\nu+1)]^2\right)A_n,$$

$$T_1^U = -\xi R T_1^{\Gamma} + \left(1 - 2\xi k_s - 2\frac{\xi[2-(1-\mu^2)k_s]}{(1-\mu^2)} + 4\frac{\xi}{1-\mu^2}\right)T_1,$$

$$C^U = -\xi R C^{\Gamma} + C_p - \frac{R^2}{Eh}[1+\xi k_s(1+\mu)]C_q.$$

We obtain the tangential displacement u_γ by utilizing its relation to the variable $U(\mathbf{x};\mathbf{x}')$ and which yields

$$u_\gamma = \frac{\mathrm{d}}{\mathrm{d}\gamma}U(\mathbf{x};\mathbf{x}') = -2\ \mathrm{Re}\ (A_n^U P_\nu^1(-\cos\gamma)) + T_1^U\left[\frac{\cos\gamma\sin\gamma}{1-\cos\gamma} - \sin\gamma\ \ln\ (1-\cos\gamma)\right] - C^U\sin\gamma \quad (39)$$

and the angular rotation β_γ from variable $\Gamma(\mathbf{x};\mathbf{x}')$ such as

$$\beta_\gamma = \frac{\mathrm{d}}{\mathrm{d}\gamma}\Gamma(\mathbf{x};\mathbf{x}') = -2\ \mathrm{Re}\ (A_n^{\Gamma} P_\nu^1(-\cos\gamma)) + T_1^{\Gamma}\left[\frac{\cos\gamma\sin\gamma}{1-\cos\gamma} - \sin\gamma\ \ln\ (1-\cos\gamma)\right] - C^{\Gamma}\sin\gamma \quad (40)$$

while due to axisymmetry, $u_\eta = \beta_\eta = 0$.

The shear resultant Q_γ is expressed in the form

$$Q_\gamma = \frac{Eh}{2(1+\mu)k_s^0}\left[\beta_\gamma + \frac{1}{R}\frac{\mathrm{d}W}{\mathrm{d}\gamma}\right] \tag{41}$$

and consequently, $Q_\eta = 0$.

The kernel expressions for the stress and moment resultants follow from (A1), (A2).

$$N_\gamma = \frac{Eh}{(1-\mu^2)R}\left[\frac{\mathrm{d}u_\gamma}{\mathrm{d}\gamma} + \mu \cot \gamma u_\gamma + (\mu+1)W\right],$$

$$N_\eta = \frac{Eh}{(1-\mu^2)R}\left[\mu\frac{\mathrm{d}u_\gamma}{\mathrm{d}\gamma} + \cot \gamma u_\gamma + (\mu+1)W\right]$$

$$M_\gamma = \frac{D}{R}\left[\frac{\mathrm{d}\beta_\gamma}{\mathrm{d}\gamma} + \mu \cot \gamma\beta_\gamma\right], \quad M_\eta = \frac{D}{R}\left[\mu\frac{\mathrm{d}\beta_\gamma}{\mathrm{d}\gamma} + \cot \gamma\beta_\gamma\right] \tag{42}$$

and

$$N_{\gamma\eta} = M_{\gamma\eta} = 0.$$

We examine the character of the fundamental singularity in the displacement and stress kernels as $\gamma \to 0$ and evaluate the predictions of the improved theory in the vicinity of the pole. It is to be expected that the differences between the classical and the improved treatments of the shell domain are most pronounced near the point of application of the singular load.

By calculating the limiting values of the Legendre functions associated with the kernels as well as of the elementary functions involved we obtain for the transverse displacement W

$$\lim_{\gamma\to 0} W = 2\left[\frac{2}{\pi}\,\mathrm{Re}\,[A_n \sin \nu\pi] - T_1\right]\lim_{\gamma\to 0} \ln\left(\sin\frac{\gamma}{2}\right) + \text{const.} \tag{43}$$

It is apparent that, while the classical theory analysis predicts a finite response of the transverse displacement W at the point of application of the singular load, according to the improved theory treatment W is unbounded, experiencing a logarithmic singularity in its kernel. This finding, however, is in complete agreement with previous results obtained in other investigations concerning both the shallow and the nonshallow approach of the shell problem.

A detailed evaluation (numerical) reveals that in the limit as $\gamma \to 0$, the displacement variables u_γ and β_γ identically satisfy

$$\lim_{\gamma\to 0}\,[u_\gamma] = 0, \quad \lim_{\gamma\to 0}\,[\beta_\gamma] = 0. \tag{44}$$

As noted earlier the above two conditions together with the equilibrium requirement around the pole expressed by the integral

$$\lim_{\gamma\to 0}\int_0^{2\pi}[Q_\gamma \cos \gamma + N_\gamma \sin \gamma]R \sin \gamma\,\mathrm{d}\eta = -1 \tag{45}$$

constitute a set of constraints that the fundamental solution must satisfy. It has also been noted that their interaction with the construction of the fundamental singularity, when set as a priori, could lead to an incorrect fundamental solution. The evaluation of the integral in (45) reveals that the condition is indeed satisfied.

The singularity encountered in the shear resultant kernel Q_γ is of the order

$$\lim_{\gamma\to 0} Q_\gamma = \frac{2Eh}{2(1+\mu)k_s^0 R}\left[\mathrm{Re}\left[A_n \frac{2}{\pi}\sin\nu\pi\right]-T_1\right]\lim_{\gamma\to 0}\left(\frac{1}{\gamma}\right). \tag{46}$$

The character of the singularities in the stress and moment resultant kernels is logarithmic, same as in the classical theory analysis, and their limiting value can be simply written as

$$\lim_{\gamma\to 0}[N_\gamma, N_\eta, M_\gamma, M_\eta] = \lim_{\gamma\to 0}\left[A_i \ln\left(\sin\frac{\gamma}{2}\right)+B_i\right] \tag{47}$$

where, A_i and B_i are constants evaluated between the limiting behavior of the Legendre functions and the respective constants associated with the Fundamental Solutions.

We should further note that the state of compression in the vicinity of the load point experienced in the classical theory, $\lim_{\gamma\to 0} N_\gamma = -\infty$, is no longer predicted by the improved theory which, in turn, demonstrates that N_γ approaches $+\infty$. Thus we conclude that, under the influence of a concentrated load applied along the outward normal, the vicinity of the pole is in tension. Such a response is expected on the grounds of physical reasoning.

SINGULAR SELF-EQUILIBRATED SOLUTIONS OF A UNIT MOMENT AND A UNIT TANGENTIAL LOAD

In order to construct singular solutions corresponding to concentrated unit moments and tangential loads, we utilize the homogeneous solutions of (1) and (2). This implies that all the components of the external force vector are zero. However, the singularities of the general solution of these equations, encountered in the kernels of the transverse displacement W and the auxiliary function Ψ, will be integrated into the solution to yield the singular states associated with the action of concentrated tangential loads and moments applied in a self-equilibrating fashion. The loading conditions, which will be reflected by the final solutions, are for the case of a moment represented by a pair of unit moments acting at π distance apart over the spherical surface and in the opposite sense. For the unit tangential load case, the solution should reflect the action of two unit tangent loads applied at two arbitrary points with π distance apart and in opposite directions. The load pair will be corrected, for equilibrium purposes, by a concentrated moment of strength $2R$ acting at the point which will be considered the south pole of the rotated geographical coordinate system.

The general solution for W and Ψ expressed in the new system (γ, η), after taking into consideration the symmetry about $\eta = 0$ in both loading cases, can be written in the form

$$W(\mathbf{x};\mathbf{x}') = [A_1 P_1^1(\cos\gamma)+A_2 Q_1^1(\cos\gamma)+B_1 P_\nu^1(\cos\gamma)+B_2 P_\nu^1(-\cos\gamma) + C_1 P_\lambda^1(\cos\gamma)+C_2 P_\lambda^1(-\cos\gamma)]\cos\eta \tag{48}$$

and

$$\Psi(\mathbf{x};\mathbf{x}') = [A'_1 P_1^1(\cos\gamma)+A'_2 Q_1^1(\cos\gamma)+B'_1 P_\omega^1(\cos\gamma)+B'_2 P_\omega^1(-\cos\gamma)]\cos\eta \tag{49}$$

where

$$\omega(\omega+1) = 2-\frac{1}{\xi k_s^0},\quad \omega = -\tfrac{1}{2}-i\rho,\quad \rho = \frac{1}{2}\left[1-4\left(2-\frac{12R^2}{k_s^0 h^2}\right)\right]^{1/2}.$$

We should note that because of the complex conjugate parameters ν and λ, the associate Legendre functions P_ν^1 and P_λ^1 are also complex conjugates. The implication of the above is that, in order for the displacement W to remain a real quantity, the arbitrary constants B_1, B_2 and C_1, C_2 are complex conjugates, respectively. In addition, since $P_1^1(\cos\gamma)$ is a regular function everywhere in the domain, its contribution to the solution of the singular system can be eliminated by setting $A_1 = A'_1 = 0$ without any loss of generality. The above is justified from the fact that the solutions retained must satisfy the singular state at $\gamma = 0$ and/or at $\gamma = \pi$.

The general expression of the auxiliary variable $\Lambda(\mathbf{x};\mathbf{x}')$ can be deduced from (28) leading to

$$\Lambda(\mathbf{x};\mathbf{x}') = C'[B'_1 P^1_\omega(\cos\gamma) + B'_2 P^1_\omega(-\cos\gamma)]\cos\eta \tag{50}$$

where $C' = -k^0_s[2-\omega(\omega+1)]$. The remaining auxiliary variables Γ and U can be derived in terms of W, Ψ and Λ. Uncoupling of the operators in eqns (A6)–(A10) yields the expressions for Γ and U variables. After the necessary manipulations we can write

$$\Gamma(\mathbf{x};\mathbf{x}') = [e_1 + e_2\nabla^2 + e_4\nabla^4]W(\mathbf{x};\mathbf{x}') + e_5 \sin\gamma \frac{\partial}{\partial\gamma}\Lambda(\mathbf{x};\mathbf{x}') \tag{51}$$

where

$$e_1 = \frac{k_1}{\alpha_\Gamma},\quad e_2 = \frac{k_2}{\alpha_\Gamma},\quad e_4 = \frac{k_4}{\alpha_\Gamma},\quad e_5 = \frac{k_5}{\alpha_\Gamma},\quad \alpha_\Gamma = R[1+2\xi k_s\mu-(1-\mu^2)\xi^2 k_s^2]$$

$$k_1 = -[1+\xi k_s(1+\mu)-2(1-\mu^2)\xi^2 k_s^3],\quad k_2 = -\xi k_s[1+\xi k_s\{2-(1-\mu^2)k_s\}]$$

$$k_4 = -\xi^2 K_s^2,\quad k_5 = \xi k^0_s R[1+\xi k_s(1+\mu)].$$

Carrying out (51) we obtain for $\Gamma(\mathbf{x};\mathbf{x}')$

$$\begin{aligned}\Gamma(\mathbf{x};\mathbf{x}') = \big(&\delta_1 A_2 Q^1_1(\cos\gamma) + \delta_2[B_1 P^1_\nu(\cos\gamma) + B_2 P^1_\nu(-\cos\gamma)] \\ &+ \delta_3[C_1 P^1_\lambda(\cos\gamma) + C_2 P^1_\lambda(-\cos\gamma)] + e_5 C'\big[\sin\gamma\ \omega(\omega+1)[B'_2 P_\omega(-\cos\gamma) \\ &- B'_1 P_\omega(\cos\gamma)] - \cos\gamma[B'_1 P^1_\omega(\cos\gamma) + B'_2 P^1_\omega(-\cos\gamma)]\big]\big)\cos\eta\end{aligned} \tag{52}$$

where

$$\delta_1 = e_1 - 2e_2 + 4e_4\pi,\quad \delta_2 = e_1 - \nu(\nu+1)e_2 + [\nu(\nu+1)]^2 e_4,$$
$$\delta_3 = e_1 - \lambda(\lambda+1)e_2 + [\lambda(\lambda+1)]^2 e_4.$$

Similarly, for the remaining auxiliary variable $U(\mathbf{x};\mathbf{x}')$ we deduce the expression

$$\begin{aligned}U(\mathbf{x};\mathbf{x}') = \Big(&e^1_U A_2 Q^1_1(\cos\gamma) + e^2_U[B_1 P^1_\nu(\cos\gamma) + B_2 P^1_\nu(-\cos\gamma)] + e_U[C_1 P^1_\lambda(\cos\gamma) \\ &+ C_2 P^1_\lambda(-\cos\gamma)] + e_0\Big[\sin\gamma\ \omega(\omega+1)[B'_2 P_\omega(-\cos\gamma) - B'_1 P_\omega(\cos\gamma)] \\ &- \cos\gamma[B'_1 P^1_\omega(\cos\gamma) + B^1_2 P^1_\omega(-\cos\gamma)]\Big] - \frac{R}{2}A'_2 \sin\gamma \frac{\mathrm{d}}{\mathrm{d}\gamma}Q^1_1\Big)\cos\eta\end{aligned} \tag{53}$$

where

$$e^1_U = -\xi R\delta_1 + 1 - 2\xi k_s - \frac{\xi[2-(1-\mu^2)k_s]}{1-\mu^2} + \frac{4\xi}{1-\mu^2}$$

$$e^2_U = -\xi R\delta_1 + 1 - 2\xi k_s - \frac{\nu(\nu+1)\xi[2-(1-\mu^2)k_s]}{1-\mu^2} + \frac{[\nu(\nu+1)]^2\xi}{1-\mu^2}$$

$$ {}_e\bar{U}^3 = \text{CONJG}\,[{}_e\bar{U}^2], \quad e_0 = \xi R_{e_5} C' + \frac{\xi RC}{2} + \frac{R}{2}. $$

The remaining components of the displacement vector u_γ and u_η are deduced from the relations

$$ u_\gamma = \frac{\partial}{\partial\gamma} U(\mathbf{x};\mathbf{x}') - R \sin\gamma \Psi(\mathbf{x};\mathbf{x}'), \quad \mu_\eta = \csc\gamma \frac{\partial}{\partial\eta} U(\mathbf{x};\mathbf{x}'), \tag{54} $$

while the angular rotations of the normal vector β_γ and β_η are derived from the expressions

$$ \beta_\gamma = \frac{\partial}{\partial\gamma}\Gamma(\mathbf{x};\mathbf{x}') - \sin\gamma\Lambda(\mathbf{x};\mathbf{x}'), \quad \beta_\eta = \csc\gamma\frac{\partial}{\partial\eta}\Gamma(\mathbf{x};\mathbf{x}'). \tag{55} $$

Substitution of the auxiliary variables into (54) leads to

$$ \begin{aligned} u_\gamma = \Bigg(& {}_e\bar{U}^1 A_2 \frac{d}{d\gamma} Q_1^1 + 2\,\text{Re}\,[{}_e\bar{U}^2\{\nu(\nu+1)[B_2 P_\nu(-\cos\gamma) - B_1 P_\nu(\cos\gamma)] \\ & -\cot\gamma[B_1 P_\nu^1(\cos\gamma) + B_2 P_\nu^1(-\cos\gamma)]\}] \\ & - \frac{e_0}{\sin\gamma}[B_1 P_\nu^1(\cos\gamma) + B_2 P_\nu^1(-\cos\gamma)] \\ & + [e_0\omega(\omega+1) - R]\sin\gamma\,[B'_1 P_\omega^1(\cos\gamma) + B'_2 P_\omega^1(-\cos\gamma)] \\ & - \frac{RA'_2}{2}\left[\sin\gamma\frac{d^2}{d\gamma^2}Q_1^1(\cos\gamma) + \cos\gamma\frac{d}{d\gamma}Q_1^1(\cos\gamma) + 2\sin\gamma Q_1^1(\cos\gamma)\right]\Bigg)\cos\eta \end{aligned} \tag{56} $$

and

$$ \begin{aligned} u_\eta &= -\csc\gamma U(\gamma)\sin\eta \\ U(\mathbf{x};\mathbf{x}') &= U(\gamma)\cos\eta. \end{aligned} \tag{57} $$

Similarly for the angular rotations we derive the expressions

$$ \begin{aligned} \beta_\gamma = \delta_1 A_2 \frac{d}{d\gamma} Q_1^1(\cos\gamma) &+ 2\,\text{Re}\,[\delta_2\{\nu(\nu+1)[B_2 P_\nu(-\cos\gamma) - B_1 P_\nu(\cos\gamma)] \\ & -\cot\gamma\,[B_1 P_\nu^1(\cos\gamma) + B_2 P_\nu^1(-\cos\gamma)]\}] \\ & + C'\left[\frac{e_5}{\sin\gamma} - \{1+\omega(\omega+1)\}\sin\gamma\right][B'_1 P_\omega^1(\cos\gamma) + B'_2 P_\omega^1(-\cos\gamma)] \end{aligned} \tag{58} $$

and

$$ \begin{aligned} \beta_\eta &= -\csc\gamma\Gamma(\gamma)\sin\eta \\ \Gamma(\mathbf{x};\mathbf{x}') &= \Gamma(\gamma)\cos\eta. \end{aligned} \tag{59} $$

Substitution of the displacement and angular rotation vector components into eqns (A1), (A2), (A3), will yield the expressions for the stress, moment and shearing stress resultants. Thus, with these last manipulations we have arrived at a general solution state of a complete sphere experiencing the effect of singularities at both of its poles ($\gamma = 0$ and $\gamma = \pi$). The character and the physical interpretation of the singularities in the kernel functions of the dependent field variables will be examined with the introduction of a limiting contour in the vicinity of the poles. This procedure will provide us a clear view as to what physical system the singularities relate to or further, to what system they could

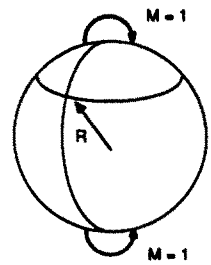

Fig. 1. Self-equilibrated moment pair.

relate to if additional limiting constraints are imposed onto the general solution of the system. The singularities will be evaluated in the form of two integrals given by (60) and (61). In these equilibrium around the two poles is implemented in the forms of a resultant tangent force and a resultant moment.

$$F_r = \lim_{\substack{\gamma\to 0\\ \gamma\to\pi}} \int_0^{2\pi} [(N_\gamma \cos\gamma + Q_\gamma \sin\gamma)\cos\eta - N_{\gamma\eta}\sin\eta] R \sin\gamma \, d\eta \tag{60}$$

$$M_r = \lim_{\substack{\gamma\to 0\\ \gamma\to\pi}} \int_0^{2\pi} R[M_\gamma \cos\eta - M_{\gamma\eta}\cos\gamma\sin\eta - R\sin\gamma\cos\eta\,[Q_\gamma\cos\gamma - N_\gamma\sin\gamma]]\sin\gamma\, d\eta. \tag{61}$$

In the two sections that follow, the construction of the singular solution states corresponding to the action of (a) a unit concentrated moment, and (b) a concentrated unit tangential force, will be formulated by utilizing the same general singular solution by being subjected to appropriate limiting conditions for each of the two cases.

Concentrated unit moment solution

We consider the self-equilibrated singular moment state shown in Fig. 1. The physical conditions needed to be imposed onto the general solution for the evaluation of the arbitrary constants are expressed in the form of the following mathematical statements:

$$W(\gamma) = W(\pi - \gamma) \tag{62a}$$

$$\lim_{\substack{\gamma\to 0\\ \gamma\to\pi}} \int_0^{2\pi} [W\cos\gamma - u_\gamma \sin\gamma + R\sin\gamma\beta_\gamma] R\sin\gamma\cos\eta\, d\eta = 0 \tag{62b}$$

$$\lim_{\substack{\gamma\to 0\\ \gamma\to\pi}} \int_0^{2\pi} [W\sin\gamma + u_\gamma\cos\gamma + u_\eta] R\sin\gamma\cos\eta\, d\eta = 0 \tag{62c}$$

$$\lim_{\substack{\gamma\to 0\\ \gamma\to\pi}} \int_0^{2\pi} [\beta_\gamma\cos\eta + \sec\gamma\beta_\eta\sin\eta] R\sin\gamma\, d\eta = 0. \tag{62d}$$

The first of the conditions, (62a), implies symmetric transverse displacement W with respect to $\gamma = \pi/2$, while through (62b) we require that the net normal displacement in the vicinity of the poles vanishes due to the fact that no net forces act along that direction. Further, the tangential displacement component in the direction of $\eta = \pm\pi/2$ should also vanish, as stated by (62c), as the two poles are approached due to the character of the applied load. With the same physical reasoning, the net angular rotation of the normal in the direction of $\eta = \pm\pi/2$ is required to vanish. This is expressed in the form of the limiting integral in (62d). In addition to relations (62) and according to (61), the net resultant moment in the direction of $\eta = 0$ of the internal stresses around a limiting contour, must balance the external couple applied at each pole. It is apparent that enough mathematical relations have been generated in order to evaluate the arbitrary constants of the singular solution.

The first of the requirements leads to the condition

$$A_2^m = 0, \tag{63a}$$

while evaluation of the limiting integrals in (62b–d), by incorporating the singular behavior of the Legendre functions, leads to the following system of equations.

$$s(\nu)B_2^m + s(\lambda)C_2^m = 0 \tag{63b}$$

$$\epsilon_U^2 s(\nu)B_2^m + \epsilon_U^3 s(\lambda)C_2^m + \epsilon_0 s(\omega)(B'_2)^m + \frac{R}{2}(A'_2)^m = 0 \tag{63c}$$

$$\delta_2 s(\nu)B_2^m + \delta_3 s(\lambda)C_2^m - \epsilon_5 (C')^m s(\omega)(B'_2)^m = 0 \tag{63d}$$

$$s(\nu)B_1^m + s(\lambda)C_1^m = 0 \tag{63e}$$

$$\epsilon_U^2 s(\nu)B_1^m + \epsilon_U^3 s(\lambda)C_1^m - \epsilon_0 s(\omega)(B'_1)^m + \frac{R}{2}(A'_2)^m = 0 \tag{63f}$$

$$\delta_2 s(\nu)B_1^m + \delta_3 s(\lambda)C_1^m + \epsilon_5 (C')^m s(\omega)(B'_1)^m = 0 \tag{63g}$$

where the superscript m on the constants denotes their affiliation with the moment solution while $s(\nu) = (2/\pi)\sin\nu\pi$ etc.

The moment requirements, evaluated around $\gamma = 0$, provide the following additional relation between the arbitrary constants.

$$\delta_2 T(\nu)B_2^m + \delta_3 T(\lambda)C_2^m + C'\left[\epsilon_5 T(\omega) + \frac{1+\mu}{2}s(\omega)\right](B'_2)^m = -\frac{1}{\pi D} \tag{64}$$

where $T(\nu) = \nu(\nu+1)(2/\pi)\sin\nu\pi$.

Although the same integral must be evaluated at $\gamma = \pi$, the requirement $W(\gamma) = W(\pi-\gamma)$ implies that

$$B_1^m = B_2^m, \quad C_1^m = C_2^m \tag{65}$$

and consequently no additional relation need be incorporated. However, when such an integral is formed, the moment condition at $\gamma = \pi$ is identically satisfied.

The system of equations generated is sufficient and so, when solved, the arbitrary constants are evaluated. Their particular values reflect the solution of the physical system described in Fig. 1. The symmetry of the system about $\gamma = \pi/2$ also yields the following relation.

$$(B'_2)^m = -(B'_1)^m. \tag{66}$$

Concentrated unit tangential load solution

By utilizing the same general solution and using the same physical requirements, (62b–d), used in the construction of the moment solution, except for (62a), we arrive at the following set of equations between the constants

$$A'_2 + s(\nu)B'_2 + s(\lambda)C'_2 = 0 \tag{67a}$$

$$A'_2 - s(\nu)B'_1 - s(\lambda)C'_1 = 0 \tag{67b}$$

$$\epsilon_U^1 A'_2 + \epsilon_U^2 s(\nu)B'_2 + \epsilon_U^3 s(\lambda)C'_2 + \epsilon^0 s(\omega)(B'_2)' + \frac{R}{2}(A'_2)' = 0 \tag{67c}$$

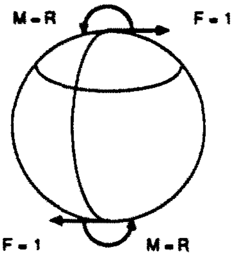

Fig. 2. Sphere in equilibrium.

$$-\epsilon_U^1 A_2' + \epsilon_U^2 s(\nu) B_1' + \epsilon_U^3 s(\lambda) C_1' - \epsilon^0(\omega)(B_1')' + \frac{R}{2}(A_2')' = 0 \tag{67d}$$

$$\delta_1 A_2' + \delta_2 s(\nu) B_2' + \delta_3 s(\lambda) C_2' - e_5 C' s(\omega)(B_2')' = 0 \tag{67e}$$

$$\delta_1 A_2' - \delta_2 s(\nu) B_1' - \delta_3 s(\lambda) C_1' - e_5 C' s(\omega)(B_1')' = 0. \tag{67f}$$

Evaluation of the force resultant, through (60), around the pole $\gamma = 0$ leads to the relation

$$2\epsilon_U^1 A_2' + \epsilon_U^2 T(\nu) B_2' + \epsilon_U^3 T(\lambda) C_2' - \left[\epsilon_0 T(\omega) - \frac{1+\mu}{2} Rs(\omega)\right](B_2')' + \frac{\mu R}{2}(A_2')' = -\frac{1-\mu^2}{Eh\pi}. \tag{68}$$

It is apparent that one additional condition is needed in order for the system of equations to become complete. However, our desired system must incorporate two additional singular effects, such as a resultant force $F = 1$ with direction opposite to the one already considered at $\gamma = 0$, and a resultant moment $M = 2R$, both applied at $\gamma = \pi$. By using the complete sphere, we suspect that overall equilibrium could be satisfied by the solution itself. Thus we proceed by calculating the moment integral at $\gamma = 0$ and letting it correspond to a resultant moment of intensity $M = R$ with direction similar to the one in Fig. 2. Such a condition leads to the relation

$$2\delta_1 A_2' + \delta_2 T(\nu) B_2' + \delta_3 T(\lambda) C_2' - C'\left[e_5 T(\omega) + \frac{1+\mu}{2} Rs(\omega)\right](B_2')' = -\frac{1-\mu^2}{Eh\pi}. \tag{69}$$

The system of eight equations is solved in terms of the arbitrary constants and subsequently the two integrals, the resultant force and the resultant moment, at $\gamma = \pi$ are evaluated. The above test indeed proves that the singular solution derived, in satisfaction of the choice of limiting conditions, describes the physical system of Fig. 2. We further note that a moment solution needs to be superimposed onto the system just derived in order to achieve the solution state of Fig. 3. However, since such a moment solution has already been derived in the form of a self-equilibrated pair of unit moments, the superposition approach requires

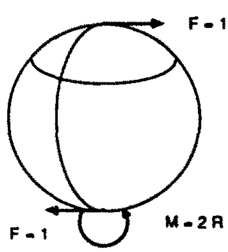

Fig. 3. A unit lateral force equilibrated by a similar force and a moment.

no additional formulation. Thus the kernel functions associated with a unit tangential load applied at an arbitrary point $x'(\phi', \theta')$, or else in the introduced rotated system (γ, η) at $\gamma = 0$ and directed along $\eta = 0$, will be of the form

$$
\begin{aligned}
W^T(\mathbf{x};\mathbf{x}') &= W^t(\mathbf{x};\mathbf{x}') + RW^m(\mathbf{x};\mathbf{x}') \\
u_\phi^T(\mathbf{x};\mathbf{x}') &= u_\phi^t(\mathbf{x};\mathbf{x}') + Ru_\phi^m(\mathbf{x};\mathbf{x}') \\
N_\phi^T(\mathbf{x};\mathbf{x}') &= N_\phi^t(\mathbf{x};\mathbf{x}') + RN_\phi^m(\mathbf{x};\mathbf{x}')
\end{aligned}
\tag{70}
$$

etc. In (70) kernel functions with superscript T (W^T, M_ϕ^T etc.) refer to the singular solutions associated with a unit tangential load. These solutions were constructed from the kernels which resulted from the general solution by utilizing the arbitrary constants with superscript t (B_2^t, A_2^t etc) and the moment kernels multiplied by R necessary to eliminate the effect of the concentrated moment of intensity R, at $\gamma = 0$, accompanying the solution of Fig. 2. With these last manipulations we have arrived at the complete set of singular solutions for all three types of concentrated loads applied on the middle surface of a closed sphere by incorporating into their expressions the effect of transverse shear deformation.

Since the differences between the classical and the improved theories are expected to occur in the vicinity of the point of application of the singular load, we evaluate the behavior of the dependent variables associated with the last two types of loading in the limit as $\gamma \to 0$. The character of the singularities experienced by the dependent variables as $\gamma \to 0$, is the same for both moment and tangential load solutions. In terms of the displacement vector components we observe similar characteristics to those of the classical theory analysis. Thus, the transverse displacement W vanishes as we approach the point of application, $\lim_{\gamma \to 0} W = 0$, while both tangential displacement components, u_γ and u_η, are unbounded as a result of logarithmic singularities in their kernels. The same is true for the angular displacements β_γ and β_η which also present logarithmic behavior in the pole vicinity.

The stress and moment resultant components also match the behavior of their counterparts in the classical analysis. All six components present singularities of order $(1/\gamma)$ and some of their limiting values are recorded below.

$$
\begin{aligned}
\lim_{\gamma \to 0} N_\gamma^m = \frac{Eh}{(1-\mu^2)R} \Bigg(&(1+\mu)\frac{R}{2}(A_2^t)^m + (1-\mu)[2\ \mathrm{Re}\ \{\epsilon_U^2 B_2^m T(v)[2CP_1(v) - CP_0(v)]\}] \\
&+ 2\ \mathrm{Re}\ [\epsilon_U^2 B_2^m T(v)] + (1-\mu)[\epsilon_0 (B_2^t)^m T(\omega)[2CP_1(\omega) - CP_0(v)]] \\
&- \mu[\epsilon_0 \omega(\omega+1) - R] s(\omega)(B_2^t)^m \Bigg) \lim_{\gamma \to 0} \left(\frac{1}{\gamma}\right) \\
\lim_{\gamma \to 0} M_\gamma^m = \frac{D}{R} \Bigg(&(1-\mu)[2\ \mathrm{Re}\ \{\delta_2 B_2^m T(v)[2CP_1(v) - CP_0(v)]\}] + 2\ \mathrm{Re}\ [\delta_2 B_2^m T(v)] \\
&- (1-\mu)[\epsilon_5 C'(B_2^t)^m T(\omega)[2CP_1(\omega) - CP_0(\omega)]] \\
&+ \mu C'[\epsilon_5 \omega(\omega+1) + 1] s(\omega)(B_2^t)^m \Bigg) \lim_{\gamma \to 0} \left(\frac{1}{\gamma}\right)
\end{aligned}
\tag{71}
$$

where

$$
\begin{aligned}
CP_0(v) &= \psi(v+1) + C + \frac{\pi}{2}\cot v\pi - \tfrac{1}{2}\ln 2 \\
CP_1(v) &= [\pi \cot v\pi + \psi(v+2) + \psi(v) + 2C - 1 - \ln 2]/4 \\
C &= \text{Euler's constant} \\
\psi(\ldots) &= \text{logarithmic derivative of Gamma function.}
\end{aligned}
$$

Similar expressions would result for the respective resultant components of the tangential load solution. Their difference with the above expressions will be the participation of the additional constant A_2 which in the general solution accompanies the independent solution $Q_1^1(\cos\gamma)$.

The most pronounced difference between the two theories, however, was observed in the character of the singularity in the shear stress resultants Q_γ and Q_η. It was found, in the classical theory approach, that these resultants demonstrate a very strong singular behavior of order $1/\gamma^2$. In the improved theory, however, the singularity involved with the above kernels is simply logarithmic. We record below the limiting value of the shear resultant of the unit moment solution.

$$\lim_{\gamma\to 0} Q_\gamma = \frac{Eh}{2(1+\mu)k_s^0} \lim_{\gamma\to 0} [A_1^0 \cos\gamma \ln(1-\cos\gamma) + A_2^0 \ln(1-\cos\gamma) + A_3^0 \cos\gamma + A_4^0] \tag{72}$$

where

$$A_1^0 = -\frac{1}{4\pi}\epsilon_5 C'\omega(\omega+1)(B_2')^m$$

$$A_2^0 = -\tfrac{1}{2}\,\mathrm{Re}\left\{\left[\delta_2+\frac{1}{R}\right]T(v)B_2^m\right\} - \frac{1}{\pi}\epsilon_5 C'\omega(\omega+1)(B_2')^m$$

$$A_3^0 = -\frac{1}{\pi}\epsilon_5 C'(B_2')^m\omega(\omega+1)CP_1(\omega)$$

$$A_4^0 = -2\,\mathrm{Re}\left\{\left[\delta_2+\frac{1}{R}\right]T(v)CP_1(v)B_2^m\right\} + \frac{1}{\pi}\epsilon_5 C'CP_0(\omega)\omega(\omega+1)(B_2')^m$$

while a similar expression results for the limiting value of Q_η.

Further, we should note that the character and strength of the singularities in all the dependent variable kernels agrees with the singular behavior previously obtained by Wilkinson and Kalnins (1966a, b). The weakness of the singularity in the shear resultants is expected to have great impact in the evaluation of physical problems where integration of the kernels over singular points is vital to the analysis.

NUMERICAL RESULTS AND DISCUSSION

The analytical solution of this paper that resulted in closed form expressions for the dependent variables of the spherical shell problem, served two purposes. First it made possible an explicit representation of the behavior of the shell in the vicinity of the point of application of singular surface loads. The effect of the shear deformation was reflected in the shell response after comparative evaluation was performed against the classical theory solution. Second, it provided formulae for the displacement and stress variables suitable for numerical calculations with the use of integral equations.

We focus our attention on the bending region which extends to some distance around the point of application of the singular load. We expect that the shell response characteristics will clearly demonstrate the effect of the shear deformability which is present only in the improved theory analysis. For that purpose a spherical shell with a ratio of thickness h to the radius R, $h/R = 1/20$ is considered. In the process of studying the effect of shear deformability greater ratios of h to R were considered. It has been observed that the differences occurring in the different field variables (such as displacements and stress resultants) increase as the shell gets thicker. However, because both theories have been formulated upon the assumption that the shell is "thin", conclusions that are solely based upon the shell thickness could be misleading and inaccurate. Although results obtained through the 3-D finite element model validated the effect of the transverse shear, as captured

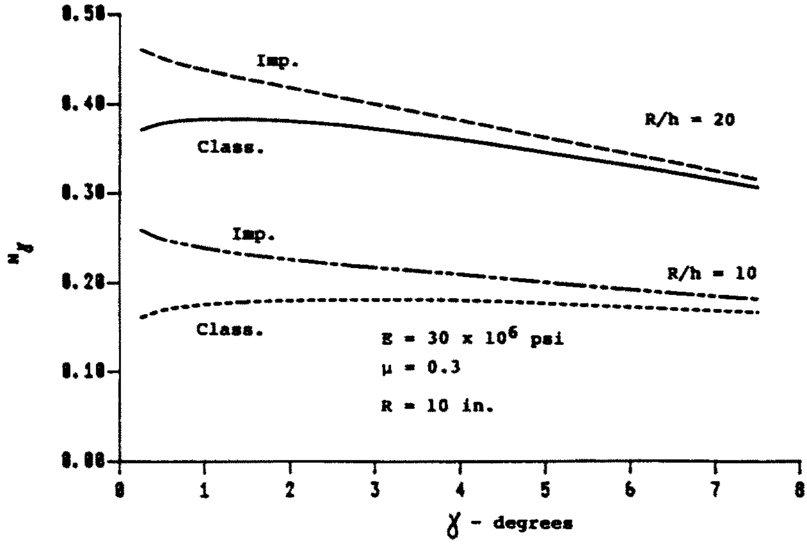

Fig. 4. Stress resultant produced by an outward going unit point load. Classical and Improved theory comparison.

by the improved theory, are left out simply because of the fully analytical character of the present solution.

Results in the bending region are presented for a normal concentrated load, a tangential load and a concentrated moment. In the first case (see Fig. 4) our analysis demonstrates the striking difference between the two theories in the stress resultant around the point of application of a unit point load acting in the outward direction. The classical theory predicts that the point load vicinity is in a state of compression while the improved theory implies that the region is in tension. While both theories have the same order of singularity in the corresponding kernels, logarithmic, the discrepancy is the result of the mathematical model that the classical theory is based on. Such model requires that the normals remain normals after deformation and for such an assumption to be satisfied compressive forces have to act in the vicinity of the point load. Similar results are presented for a thicker shell, $h/R = 1/10$, and it is apparent that the bending region extends further as a result of the shear deformability.

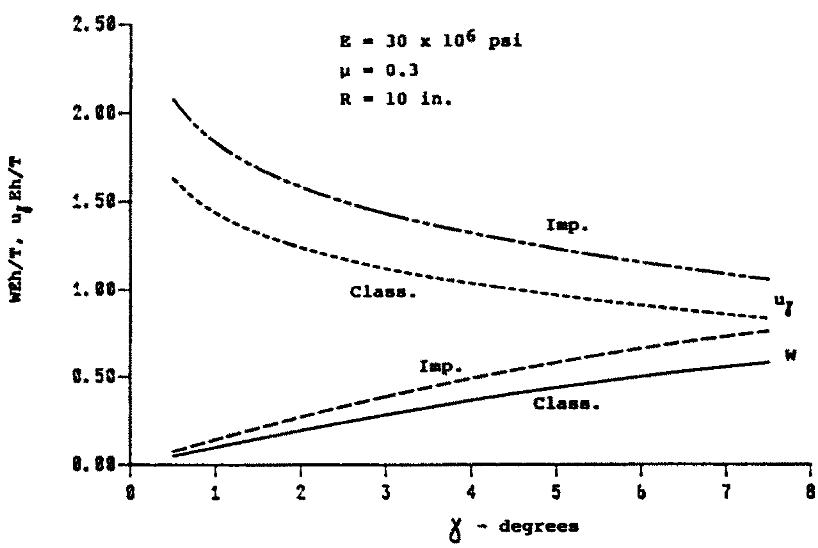

Fig. 5(a). Displacement solution resulted by the action of a unit tangential load. Classical and Improved theory comparison.

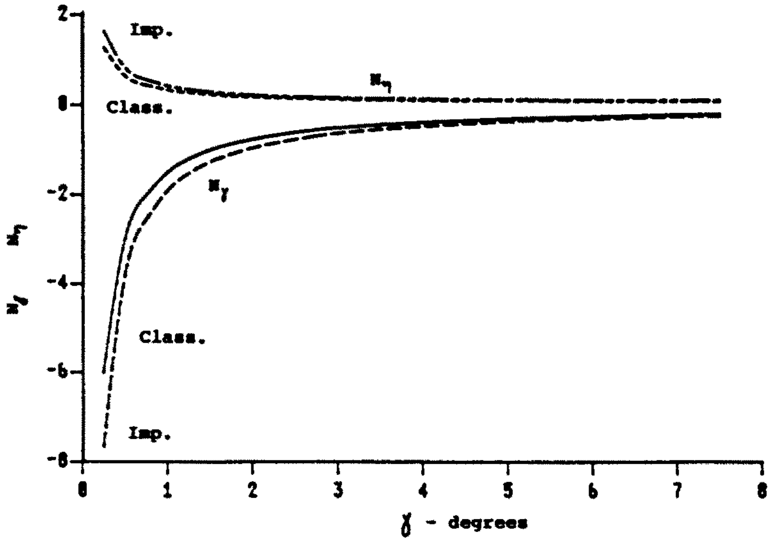

Fig. 5(b). Stress resultants due to the action of a unit tangential load.

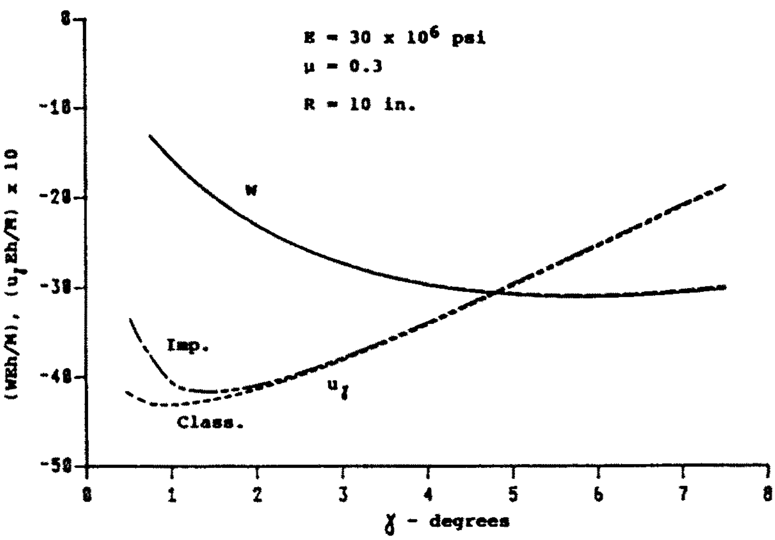

Fig. 6. Displacement solution in the vicinity of a concentrated unit moment. Classical and Improved theory comparison.

Next we look at the state of deformation and stress in the bending region of a shell acted upon by a unit tangential load. While the singular character of the associated kernels of both theories are of the same order the effect of the transverse shear is evident. In Fig. 5(a) we present the shell deformation which suggests that the bending region extends further than in the case of a normally applied load. It is also interesting to note that the shell deformation seems to be more sensitive to the shear deformability than the stress resultants, shown in Fig. 5(b), at some distance from the applied load. Finally the deformation of the vicinity of a concentrated moment is presented in Fig. 6. We should also note that while the membrane-inextensional solution, which comes about from the operator ∇^2+2, is the same for both approaches, the differences arise in the rapidly varying parts of the solutions which are expressed in the forms of the Legendre functions.

On a final note, when the singular solutions that have been obtained in closed form are utilized in boundary integral approaches to solve shell problems with boundaries, noticable discrepancies will arise over boundaries with free edge conditions as well as in their vicinity. That is in account of only four boundary conditions that the classical theory incorporates, using the effective Kirchhoff resultants, while the improved analysis allows five independent constraints along the shell boundary.

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APPENDIX A

The various stress resultants of the shell element in the (ϕ, θ) surface coordinates are expressed in terms of the surface displacement and the angular displacement vectors:

$$N_\phi = \frac{Eh}{(1-\mu^2)R}\left[\frac{\partial u_\phi}{\partial \phi} + W + \mu\left(u_\phi \cot\phi + \csc\phi \frac{\partial u_\theta}{\partial \theta} + W\right)\right]$$

$$N_\theta = \frac{Eh}{(1-\mu^2)R}\left[\mu\left(\frac{\partial u_\phi}{\partial \phi} + W\right) + u_\phi \cot\phi + \csc\phi \frac{\partial u_\theta}{\partial \theta} + W\right]$$

$$N_{\phi\theta} = \frac{Eh}{2(1+\mu)R}\left[\frac{\partial u_\phi}{\partial \theta}\csc\phi + \frac{\partial u_\theta}{\partial \phi} - \cot\phi u_\theta\right] \tag{A1}$$

$$M_\phi = \frac{D}{R}\left[\frac{\partial \beta_\phi}{\partial \phi} + \mu\left(\frac{\partial \beta_\theta}{\partial \theta}\csc\phi + \cot\phi \beta_\phi\right)\right]$$

$$M_\theta = \frac{D}{R}\left[\mu \frac{\partial \beta_\phi}{\partial \phi} + \frac{\partial \beta_\theta}{\partial \theta}\csc\phi + \cot\phi \beta_\phi\right]$$

$$M_{\phi\theta} = \frac{(1-\mu)D}{2R}\left[\frac{\partial \beta_\theta}{\partial \phi} + \frac{\partial \beta_\phi}{\partial \theta}\csc\phi - \beta_\theta \cot\phi\right] \tag{A2}$$

$$Q_\phi = \frac{Eh}{2(1+\mu)k_t^0}\left[\beta_\phi + \frac{1}{R}\frac{\partial W}{\partial \phi}\right]$$

$$Q_\theta = \frac{Eh}{2(1+\mu)k_t^0}\left[\beta_\theta + \frac{1}{R\sin\phi}\frac{\partial W}{\partial \theta}\right]. \tag{A3}$$

Prasad's (1964) auxiliary variables relate to the displacement vectors according to the expressions

$$u_\phi = \frac{\partial U}{\partial \phi} - R\Psi \sin\phi, \quad u_\theta = \frac{\partial U}{\partial \theta}\csc\phi \tag{A4}$$

$$\beta_\phi = \frac{\partial \Gamma}{\partial \phi} - \Lambda \sin\phi, \quad \beta_\theta = \frac{\partial \Gamma}{\partial \theta}\csc\phi. \tag{A5}$$

Secondary system of equations

$$(\nabla^2+1-\mu)U-\left[\frac{(1+\mu)}{2}\frac{\partial\Psi}{\partial\phi}R\sin\phi+2\Psi R\cos\phi\right]+\left(1+\mu+\frac{1}{k}\right)W+\frac{R}{k_s}\Gamma=0 \tag{A6}$$

$$(\nabla^2+2)\Psi+\frac{1}{k_s^0}\Lambda=0. \tag{A7}$$

$$\left[\nabla^2+1-\mu-\frac{1}{\xi k_s}\right]\Gamma-\left[\frac{(1+\mu)}{2}\frac{\partial\Lambda}{\partial\phi}\sin\phi+2\Lambda\cos\phi\right]-\frac{W}{ERk_s}=0 \tag{A8}$$

$$\left(\nabla^2+2-\frac{1}{\xi k_s^0}\right)\Lambda=0. \tag{A9}$$

$$R\nabla^2\Gamma-(1+\mu)k_s\nabla^2U+[\nabla^2-2(1+\mu)k_s]W-R\left(2\cos\phi+\sin\phi\frac{\partial}{\partial\phi}\right)[\Lambda-(1+\mu)k_s\Psi]+\frac{(1-\mu^2)R^2}{Eh}k_sq_n=0. \tag{A10}$$